Matrices Associated with Graphs

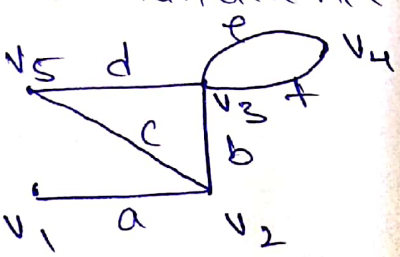
I. Incidence matrix : (I m)

Let G be a graph with 'n' vertices $v_1, v_2, \ldots, v_n$ and 'm' edges $e_1, e_2, \ldots, e_m$ and no self - loops. Then the incidence matrix $A = [a_{ij}]_{n \times m}$ is defined as—

$$a_{ij} = \begin{cases} 1, & \text{if } j^{th} \text{ edge } e_j \text{ is incident on } i^{th} \text{ vertex } v \\ 0, & \text{otherwise} \end{cases}$$

Any element of I M is either 0 or 1, ∴ it is also called (0,1) matrix or a bit matrix.

Ex:- To formulate incidence matrix for the graph →



5 vertices
6 edges
∴ (5×6) matrix

$$Im = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Properties of IM

1. Each colm has exactly two 1's because each edge is incident on exactly two vertices
2. The sum of each row gives the degree of the corresponding vertex
3. If a graph has // edges then corresponding colms are identical
4. A row with all zero's represents an isolated vertex
5. From an IM, the corresponding graph can be easily drawn.
6. If a graph G is disconnected and H_1 & H_2 are its components then the incidence matrix A(G) of G can be written as

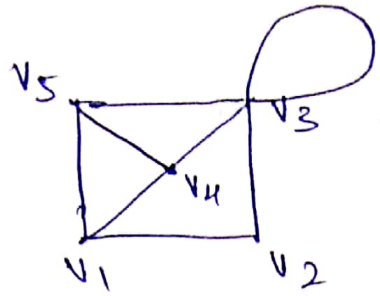
$$A(G) = \left[\begin{array}{c|c} A(H_1) & 0 \\ \hline 0 & A(H_2) \end{array}\right]$$

II. Adjacency matrix (AM)

AM of a graph with n vertices and no // edges (here self-loops are allowed) is an $n \times n$ matrix $X = [x_{ij}]$ defined as

$$x_{ij} = \begin{cases} 1, & \text{if there is an edge b/w } i^{th} \text{ \& } j^{th} \text{ vertices} \\ 0, & \text{if there is no edge b/w them} \end{cases}$$

Ex- Formulate AM for the graph -



$n = 5 \rightarrow$ no. of vertices

$\therefore$ 5×5 matrix

Soln -

$$X = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Properties of AM

1. It is symmetric i.e. $X' = X$
2. If the graph has no self-loops then the diagonal entries of AM are zero and vice-versa
3. If the graph has no self-loops then sum of 'i^{th}' row is the degree of vertex 'V_i'
4. Two graphs G_1 & G_2 are isomorphic iff the AM of one can be obtained from the AM of the others by interchanging some of the rows and the corresponding colms.
5. If G has two components H_1 & H_2 then the AM of G is -

$$X(G) = \left[\begin{array}{c|c} X(H_1) & 0 \\ \hline 0 & X(H_2) \end{array} \right]$$